Assignment 1

Exercise 1. Let C be a code with minimum distance d. Prove that C can correct any pattern of e_1 errors and e_2 erasures provided that $2e_1 + e_2 + 1 \leq d$. (Hint: given an erasure pattern, consider the code obtained by the deleting the erasure positions.)

Exercise 2 (RAID, distributed storage)**.** Redundant Arrays of Independent Disks consist of a set of disks such that any subset of s disks can be disabled and the others are still able to reconstruct any requested file (the system can tell which disks are disabled).The rate of a RAID system corresponds to the rate at which data is stored.

1. Design a RAID system for 7 disks and $s = 2$. To do this you may want to consider a code of length 7 which has minimum distance d such that s erasures can be corrected, for instance, the $(7, 4)$ Hamming code which has $d = 3$.
2. What happens if we use this code and try correct 3 erasures?

Exercise 3 (Best decoder)**.** Consider a set of $\mathcal{M}$ messages. A random message M is chosen with probability $P(M = m) = p_m$ (hence $\sum_m p_m = 1$), encoded, and sent across a channel. Upon observing the channel output y, the receiver declares one of the messages by means of a decoder which maps each channel output to one of the messages. Let D^* be the Maximum A Posteriori (MAP) decoding rule, i.e.

$$D^*(y) = \arg\max_m P(m|y).$$

1. Show that among all decoding functions, D^* minimizes the error probability given any channel output.
2. Deduce that D^* minimizes the average error probability among all decoding function.

Exercise 4 (MAP decoder)**.** Consider communication over a binary symmetric channel with crossover probability p. There are two possible equally likely messages that are encoded over three bits: 000 and 111. What is the error probability of the MAP decoder?

Exercise 5 ($A(n, d, w), A(n, d)$)**.** For any integers n, d, w let $A(n, d, w)$ be the largest possible size of a set of binary vectors of length n and weight w whose minimum distance is at least d, and let $A(n, d)$ be the largest possible size of a set of length n binary vectors whose minimum distance is at least d. Prove that

$$A(n, d) \leq \sum_{w=0}^{n} A(n, d, w)$$