Addenda to instantaneous codes

September 2023

Suppose we are given a set of codeword finite lengths $\{l_x\}_{x\in\mathcal{X}}$ that satisfy the Kraft inequality

$$\sum_{x\in\mathcal{X}} 2^{-l_x} \leq 1.$$

We now show that with these lengths one can construct an instantaneous code.

First, let us relabel the lengths over $\mathbf{N}$ and assume without loss of generality that

$$l_1 \leq l_2 \leq \ldots \leq l_{|\mathcal{X}|} < \infty.$$

Now, let $n_1 = 0$ and define

$$n_j \triangleq \sum_{i=1}^{j-1} 2^{-l_i} \quad j = 2, \ldots, |\mathcal{X}|$$

Because of Kraft inequality, we deduce that

$$n_j < 1 \quad j = 1, \ldots, |\mathcal{X}|$$

Now define codeword $c(j) \in \{0,1\}^*$ as the first l_j bits of the binary expansion of number n_j. For example, if $l_1 = 1, l_2 = l_3 = 2$ we have $c(1) = 0$, $c(2) = 10$ (the first two bits of the binary expansion of $2^{-1} = 1 \times 2^{-1} + 0 \times 2^{-2}$), and $c(3) = 11$ (the first 2 bits of the binary expansion of $2^{-1} + 2^{-2} = 1 \times 2^{-1} + 1 \times 2^{-2}$)

We prove that c is a prefix-free code by contradiction. Consider two indices $s < t$. Notice that by assumption $l_s \leq l_t$. Suppose that $c(s)$ is the prefix of $c(t)$. By construction this implies that $n_t - n_s \leq 2^{-l_s}$ since n_t and n_s agree on the most significant l_s bits. However,

$$n_t - n_s = 2^{-l_s} + 2^{-l_{s+1}} + \ldots > 2^{-l_s},$$

which is a contradiction.