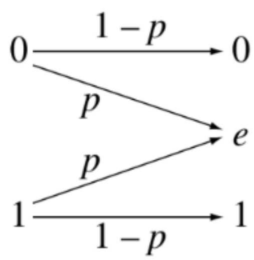
ASSIGNMENT 4

Exercise 1 (Binary erasure channel)**.** A binary erasure channel with erasure probability β, denoted BEC(p) has output alphabet $\{0, 1, e\}$ and transitions given by $P(0|0) = P(1|1) = 1-p$, and $P(e|0) = P(e|1) = p$ where e is the erasure symbol.



a. Show that the capacity of BEC(p) is $1 - p$.

Solution. We have

$$\begin{aligned} P(Y = e) &= P(Y = e, X = 0) + P(Y = e, X = 1) \\ &= P(X = 0)p + P(X = 1)p \\ &= p. \end{aligned}$$

Since

$$\begin{aligned} P(X = 0|Y = e) &= \frac{P(Y = e|X = 0)P(X = 0)}{P(Y = e)} \\ &= \frac{pP(X = 0)}{p} \\ &= P(X = 0), \end{aligned}$$

we have $H(X|Y = e) = H(X)$. Observe that

$$H(X|Y = 0) = H(X|Y = 1) = 0.$$

Hence,

$$\begin{aligned} I(X; Y) &= H(X) - H(X|Y) \\ &= (1 - p)H(X), \end{aligned}$$

which is maximized when $X \sim Ber(1/2)$ yielding the result. □

b. Suppose that the message W is chosen from $\{0,1\}$ uniformly and that perfect feedback is available at the transimtter. This means that he n-th codeword symbol for a message W is a function of W and the feedback, given by

$$X_n = X_n(W, Y^{n-1}).$$

We use the following code: to send $W = 0$, keep sending 0 until it is received without being erased. The decodes outputs an estimate of the message at time T where

$$T = \inf\{n : Y_n \neq e\}.$$

The decoder uses a rule $g(Y_1, \ldots, Y_T)$. What is the optimal rule?

Solution. The optimal rule is

$$g(Y_1, \ldots, Y_T) = Y_T,$$

and the probability of error is 0. □

c. Since the number of transmissions per message is variable, we define the rate of the code to be

$$R = \frac{\log \mathcal{M}}{\mathbb{E}T},$$

where $\mathbb{E}T$ is the expected value of T. What is the rate of the aforementioned code?

Solution.

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{P}[W=0]\mathbb{E}[T|W=0] + \mathbb{P}[W=1]\mathbb{E}[T|W=1] \\
&= \frac{1}{2}\mathbb{E}[T|W=0] + \frac{1}{2}\mathbb{E}[T|W=1] \\
&= \frac{1}{1-p},
\end{aligned}$$

whereby $R = \frac{1}{\frac{1}{1-p}} = 1 - p$. □

d. Calculate the probability of error.

Solution. The probability of error is 0 since $g(Y_T) = Y_T = X_1 = M$. □

Exercise 2. a. Suppose that the transmitter sends X through a channel and the receiver receives Y. What is the decoding rule g that minimizes $\mathbb{P}[g(Y) \neq X]$?

Solution. The probability of error is given by

$$\begin{aligned}
\mathbb{P}[g(Y) \neq X] &= \sum_{x,y} \mathbb{P}[X = x, Y = y]\mathbf{1}_{\{g(y)\neq x\}} \\
&= \sum_{x,y} P_X(x)P_{Y|X}(y|x)\mathbf{1}_{\{g(y)\neq x\}}
\end{aligned}$$

Let $g(y) = x^*$ where $x^* = \arg\max P_X(x)P_{Y|X}(y|x)$. □

b. Suppose that one of the codewords $\{0000, 1111\}$, each of which is equally likely, is sent over a BSC(p). What is the optimal decoding rule in this case?

Proof. The optimal decoding rule is to output the codeword that is closest in Hamming distance to the received codeword. □

Exercise 3 (Z-channel)**.** The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by:

$$p(0|0) = 1 - p(1|0) = 1,$$
$$p(0|1) = p(1|1) = 1/2.$$

Find the capacity of the Z-channel and the maximizing input probability distribution.

Solution. Let $p = \mathbb{P}(X = 1)$. So, $\mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = 0) = \frac{p}{2}$

$$H(Y|X) = H(Y|X = 0)\mathbb{P}(X = 0) + H(Y|X = 1)\mathbb{P}(X = 1) = 0 + 1 \cdot p = p$$

$$H(Y) = h_b\left(\frac{p}{2}\right)$$

$$I(X;Y) = H(Y) - H(Y|X) = h_b\left(\frac{p}{2}\right) - p$$

Taking derivative with respect to p, it can be seen that the mutual information gets maximized for $p = \frac{2}{5}$ and the capacity is thus $C \simeq 0.32$. □

Exercise 4 (Binary multiplier channel)**.** Consider the channel $Y = X \cdot Z$, where X and Z are independent binary random variables and $Z \sim Ber(\alpha)$. [i.e., $P(Z = 1) = \alpha$].

a. Find the capacity of this channel and the maximizing distribution on X.

b. Now suppose that the receiver can observe Z as well as Y. What is the capacity?

Solution. a. The transition probability matrix is

$$Q = \begin{pmatrix} 1 & 0 \\ 1 - \alpha & \alpha \end{pmatrix}$$

So, if $\mathbb{P}(X = 1) = p$, then

$$H(Y|X) = H(Y|X = 0)\mathbb{P}(X = 0) + H(Y|X = 1)\mathbb{P}(X = 1) = 0 \cdot (1 - p) + h_b(\alpha) \cdot p = p h_b(\alpha).$$

$$H(Y) = h_b(\alpha \cdot p)$$

$$I(X;Y) = H(Y) - H(Y|X) = h_b(\alpha \cdot p) - p h_b(\alpha)$$

Taking derivative with respect to p and let it equal to zero, we have

$$\alpha \log\left(\frac{1 - \alpha p^*}{\alpha p^*}\right) = h_b(\alpha),$$

$$p^* = \frac{1}{\alpha\left(2^{\frac{h_b(\alpha)}{\alpha}} + 1\right)}.$$

and

$$
\begin{aligned}
C = I(X;Y)|_{p=p^*} &= h_b(\alpha \cdot p^*) - p^* h_b(\alpha) \\
&= p^*\alpha \log\left(\frac{1-\alpha p^*}{\alpha p^*}\right) - \log(1-\alpha p^*) - p^* h_b(\alpha) \\
&= -\log(1-\alpha p^*) \\
&= \log\left(\frac{2^{\frac{h_b(\alpha)}{\alpha}}+1}{2^{\frac{h_b(\alpha)}{\alpha}}}\right)
\end{aligned}
$$

Note that for $\alpha = \frac{1}{2}$, this channel is the same as Z-channel.

b. In this case,

$$C = \max_{p(x)} I(X;Y,Z) = \max_{p(x)}[I(X;Z) + I(X;Y|Z)] = \max_{p(x)} I(X;Y|Z).$$

Assuming $\mathbb{P}(X=1) = p$,

$$H(Y|Z) = H(Y|Z=0)Pr(Z=0) + H(Y|Z=1)Pr(Z=1) = 0\cdot(1-\alpha) + H(X|Z=1)\cdot\alpha = \alpha H(X) = \alpha h_b(p)$$

$$H(Y|X,Z) = 0$$

$$I(X;Y|Z) = H(Y|Z) - H(Y|X,Z) = \alpha h_b(p)$$

which is maximized for $p = \frac{1}{2}$ and hence, $C = \alpha$.

□

Exercise 5 (Unused symbols)**.** Show that the capacity of the channel with transition probability matrix

$$Q = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

is achieved by a distribution that places zero probability on one of input symbols. What is the capacity of this channel?

Solution. Let $P_X = (p_1, p_2, p_3)$ be a capacity-achieving distribution. Then,

$$
\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) \\
&= H\left(\frac{2}{3}p_1 + \frac{1}{3}p_2, \frac{1}{3}, \frac{1}{3}p_2 + \frac{2}{3}p_3\right) - (p_1+p_3)H\left(\frac{2}{3}, \frac{1}{3}\right) - p_2 \log_2 3.
\end{aligned}
$$

Substituting $p_2 = 1 - (p_1 + p_3)$, we obtain

$$I(X;Y) = H\left(\frac{1}{3} + \frac{1}{3}(p_1 - p_3), \frac{1}{3}, \frac{1}{3} + \frac{1}{3}(p_1 - p_3)\right) - (p_1+p_3)H\left(\frac{2}{3}, \frac{1}{3}\right) - (1-(p_1+p_3))\log_2 3.$$

For a fixed value of $p_1 + p_3$, the above expression is maximized when $p_1 = p_3$. Then, we have

$$
\begin{aligned}
I(X;Y) &= \log_2 3 - (p_1+p_3)H\left(\frac{2}{3}, \frac{1}{3}\right) - (1-(p_1+p_3))\log_2 3 \\
&= (p_1+p_3)\left\{\log_2 3 - H\left(\frac{2}{3}, \frac{1}{3}\right)\right\}.
\end{aligned}
$$

Since the second term in the above product is positive, $I(X;Y)$ is maximized when $p_1 + p_3$ is maximized. Thus, set $p_1 + p_3 = 1$ which implies that $P_X = (1/2, 0, 1/2)$. This yields the capacity to be

$$C = \log_2 3 - H\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3} \text{ bits.}$$

$\square$