ASSIGNMENT 3

Exercise 1 (Guessing, Huffman). There are 6 bottles of wine, one of which you know has gone bad. You do not know which bottle contains the bad wine, but you know that the probability of each bottle being bad is $(8/23, 6/23, 4/23, 2/23, 2/23, 1/23)$. The bad wine has a distinctive taste. To find the bad wine your friend suggests you to taste a little bit of each wine until you find the bad wine.

a. To have the least number of tastings on average, what should your strategy be? Which bottle should be tasted first?

b. What is the average number of tastings to find the bad wine?

c. Calculate the minimum average number of tastings if you are allowed to taste a mixture of different wines and detect a bad wine's taste inside (the distinctive taste is retained even when mixed with other good wines).

d. Is the strategy studied in (a) optimal if you are allowed to mix wines?

Exercise 2 (Entropy and Yes/No questions). We are asked to determine an object by asking yes-no questions. The object is drawn randomly from a finite set according to a certain distribution. Playing optimally, we need 38.5 questions on the average to find the object. At least how many elements does the finite set have?

Exercise 3. (Mixing increases entropy) Show that the entropy of the probability distribution, $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$ is less than that of the distribution $\left(p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_m\right)$.

Exercise 4. (Entropy of common distributions) Calculate the entropy of X where

a. X is the output of n independent tosses of a coin which shows heads with probability p.

b. X is a $Geo(p)$ random variable. That is, $\mathbb{P}[X = k] = (1-p)^{k-1}p$.

Exercise 5. (KL divergence) Calculate the KL divergence (relative entropy) between P and Q where

a. $P \equiv Geo(p)$ and $Q \equiv Geo(q)$.

b. $P \equiv \mathcal{N}(\mu_1, \sigma^2)$ and $Q \equiv \mathcal{N}(\mu_2, \sigma^2)$

Exercise 6 (Mutual information). a. Let X be a uniform random variable over $\{1, 2, 3, 4\}$. Let

$$Y = \begin{cases} 0 \text{ if } X \text{ is odd} \\ 1 \text{ otherwise.} \end{cases} \qquad Z = \begin{cases} 0 \text{ if } X \text{ is even} \\ 1 \text{ otherwise.} \end{cases}$$

Find $I(Y; Z)$.

b. We roll a fair die which has six sides (opposite sides of a die add up to 7). What is the mutual information between the top side and the one facing you?

Exercise 7 (Entropy and Mutual Information). Prove the following inequalities:

a. $H(X,Y|Z) \geq H(X|Z)$,

b. $I(X,Y;Z) \geq I(X;Z)$,

c. $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$.

Exercise 8 (Conditioning for mutual information)**.** Give examples of joint random variables X, Y, and Z such that

a. $I(X;Y|Z) < I(X;Y)$.

b. $I(X;Y|Z) > I(X;Y)$.

Exercise 9 (Entropy and pairwise independence)**.** Let X, Y, Z be three binary Bernoulli($\frac{1}{2}$) random variables that are pairwise independent; that is, $I(X;Y) = I(X;Z) = I(Y;Z) = 0$.

a. Under this constraint, what is the minimum value for $H(X,Y,Z)$?

b. Give an example achieving this minimum.

Exercise 10. (Conditioning and sub additivity) Prove the following.

a.
$$H(X_1, X_2, X_3) \leq \frac{1}{2}\left[H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)\right].$$

b.
$$H(X_1, X_2, X_3) \geq \frac{1}{2}\left[H(X_1, X_2|X_3) + H(X_2, X_3|X_1) + H(X_3, X_1|X_2)\right].$$

Exercise 11. Show that among all $\mathbb{N}$-valued random variables X with $\mathbb{E}[X] = \mu$, the $Geo(1/\mu)$ random variable has the maximum value of Shannon entropy.
Hint – Consider random variables X and Y with mean μ and taking values in $\mathbb{N}$ with $X \sim P_X$ and $Y \sim P_Y$ where P_Y is Geometric, and calculate $D(P_X||P_Y)$.

Exercise 12 (Conditional mutual information)**.** Consider a sequence of n binary random variables $X_1, X_2, \cdots, X_n$. Each sequence with an even number of 1's has probability $2^{-(n-1)}$, and each sequence with an odd number of 1's has probability 0. Find the mutual informations $I(X_1;X_2)$, $I(X_2;X_3|X_1), \ldots, I(X_{n-1};X_n|X_1,\ldots,X_{n-2})$.