

ASSIGNMENT 3

Exercise 1 (Error decomposition). Let h_S be an $\text{ERM}_{\mathcal{H}}$ predictor for some function class \mathcal{H} . Write the prediction error $L_P(h_S) = \mathbb{E}_{Z \sim P}(\ell(Z, h_S))$ as

$$L_P(h_S) = \varepsilon_{\text{app}} + \varepsilon_{\text{est}}$$

where $\varepsilon_{\text{app}} := \min_{h \in \mathcal{H}} L_P(h)$ and $\varepsilon_{\text{est}} := L_P(h_S) - \varepsilon_{\text{app}}$. Interpret this error decomposition.

Exercise 2 (VC dimension, parity). Let $\mathcal{X} = \{0, 1\}^n$. Given $\mathcal{I} \in \{1, 2, \dots, n\}$ let

$$h_{\mathcal{I}}(x) = \left(\sum_{i \in \mathcal{I}} x_i \right) \pmod{2}$$

denote the parity of x over the coordinates in \mathcal{I} . Show that the VC dimension of the set of all such functions, that is

$$\mathcal{H}_{\text{parity}} = \{h_{\mathcal{I}} : \mathcal{I} \subset \{1, 2, \dots, n\}\},$$

is n .

Exercise 3 (VC dimension, signed intervals). Consider the class of signed intervals over $\mathcal{X} = \mathbb{R}$

$$\mathcal{H} = \{h_{a,b,s} : a \leq b, s \in \{-1, 1\}\}$$

where $h_{a,b,s}(x) = s$ if $x \in [a, b]$ and $h_{a,b,s}(x) = -s$ if $x \notin [a, b]$. Show that $\text{VCdim}(\mathcal{H})=3$.

Exercise 4 (VC dimension, halfspaces). A homogeneous halfspace is specified by a vector \mathbf{w} in \mathbb{R}^d which defines a binary function

$$\mathbf{x} \mapsto h_{\mathbf{w}}(\mathbf{x}) := \text{sign}\langle \mathbf{w}, \mathbf{x} \rangle$$

Show that the VCdimension of the class of homogeneous halfspaces in \mathbb{R}^d is equal to d . Show that the VCdimension of the class of non-homogeneous halfspaces defined by

$$\mathbf{x} \mapsto h_{\mathbf{w},b}(\mathbf{x}) := \text{sign}\langle \mathbf{w}, \mathbf{x} \rangle + b$$

with \mathbf{w} in \mathbb{R}^d and b in \mathbb{R} is $d + 1$.

Exercise 5 (VC dimension, bounds). In class we established the upper bound $\text{VCdim}(\mathcal{H}) \leq \log(|\mathcal{H}|)$. Here we will show that this bound can be quite loose.

1. Find an example of a class \mathcal{H} of functions on the unit interval $[0, 1]$ such that $\text{VCdim}(\mathcal{H}) < \infty$ while $|\mathcal{H}| = \infty$.
2. Find an example of class \mathcal{H} of two functions such that $\text{VCdim}(\mathcal{H}) = 0$.