ASSIGNMENT 6

Exercise 1. (Converse for Gaussian channel $X \to X + Z$, $X \sim \mathcal{N}(0, \sigma^2)$) Consider any $(2^{nR}, n)$ code that satisfies the power constraint, that is,

$$\frac{1}{n}\sum_{i=1}^{n} x_i(w)^2 \leq P,$$

for $w = 1, 2, \ldots, 2^{nR}$. Let P_i denote the average power of the i-th column of the codebook, that is,

$$P_i = \frac{1}{2^{nR}}\sum_{w} x_i(w)^2.$$

a. Let W be distributed uniformly over $\{1, 2, \ldots, 2^{nR}\}$. Let $\widehat{W}$ be the estimate of W based on Y^n. Let $\epsilon_n \to 0$ as probability of error for the code goes to 0. Justify the steps with labels on the equality or the inequality signs.

$$\begin{aligned} nR = H(W) &= I(W;\widehat{W}) + H(W|\widehat{W}) \\ &\overset{(a)}{\leq} I(W;\widehat{W}) + n\epsilon_n \\ &\overset{(b)}{\leq} I(X^n;Y^n) + n\epsilon_n \\ &= h(Y^n) - h(Y^n|X^n) + n\epsilon_n \\ &\overset{(c)}{=} h(Y^n) - h(Z^n) + n\epsilon_n \\ &\overset{(d)}{\leq} \sum_{i=1}^{n} h(Y_i) - h(Z_i) + n\epsilon_n. \end{aligned}$$

b. Calculate $\mathbb{E}[Y_i^2]$ and deduce that

$$nR \leq \frac{1}{2}\sum_{i=1}^{n} \log\left(1 + \frac{P_i}{\sigma^2}\right) + n\epsilon_n.$$

c. Prove that $\frac{1}{n}\sum_{i=1}^{n} P_i \leq P$.

d. Use Jensen's inequality to conclude that

$$R \leq \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

for any code for which the probability of error goes to 0.

Exercise 2. (List decoding Fano's inequality) Consider discrete random variables X and Y with X taking values in the set $\{0, 1\}^k$. Upon observing Y we produce a list $L(Y)$ of size 2^ℓ such that

$$\mathbb{P}(X \in L(Y)) \geq 1 - \varepsilon.$$

Show that

$$H(X|Y) \leq \varepsilon k + (1-\varepsilon)\ell + 1.$$

Hint – Define the random variable $T = \mathbf{1}_{\{X \in L(Y)\}}$. Expand $H(X, Y, T)$ using chain rule.

Exercise 3. (Shearer's lemma) Shearer's lemma is a generalization of the basic inequality

$$H(X_1, \ldots, X_n) \leq \sum_{i=1}^{n} H(X_i).$$

For $S \subseteq [n] = \{1, 2, \ldots\}$, we write $X_S = (X_i : i \in S)$.

a. Prove the following Lemma: Let $X_1, \ldots, X_n$ be random variables. Let $S_1, \ldots, S_m \subseteq [n]$ be subsets such that each $i \in [n]$ belongs to at least k sets. Then,

$$kH(X_1, \ldots, X_n) \leq \sum_{j=1}^{m} H(X_{S_j}).$$

b. Suppose n distinct points in $\mathbb{R}^3$ have n_1 distinct projections on the XY-plane, n_2 distinct projections on the XZ-plane, and n_3 distinct projections on the YZ-plane. For two different points, since all three projections cannot be the same, we have $n \leq n_1 n_2 n_3$. Using Shearer's lemma, show that

$$n \leq \sqrt{n_1 n_2 n_3}.$$

Hint – Let $P = (X_1, X_2, X_3)$ be one of the n points picked uniformly at random. Then, $P_1 = (X_1, X_2)$, $P_2 = (X_1, X_3)$, and $P_3 = (X_2, X_3)$ are its three projections.