ASSIGNMENT 3

Exercise 1. (Mixing increases entropy) Show that the entropy of the probability distribution, $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$ is less than that of the distribution $\left(p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_m\right)$.

Exercise 2. (Entropy of common distributions) Calculate the entropy of X where

a. X is the output of n independent tosses of a coin which shows heads with probability p.

b. X is a $Geo(p)$ random variable. That is, $\mathbb{P}[X = k] = (1-p)^{k-1}p$.

Exercise 3. (KL divergence) Calculate the KL divergence (relative entropy) between P and Q where

a. $P \equiv \mathcal{N}(\mu_1, \sigma^2)$ and $Q \equiv \mathcal{N}(\mu_2, \sigma^2)$

a. $P \equiv Geo(p)$ and $Q \equiv Geo(q)$.

Exercise 4 (Mutual information)**.** a. Let X be a uniform random variable over $\{1, 2, 3, 4\}$. Let

$$Y = \begin{cases} 0 \text{ if } X \text{ is odd} \\ 1 \text{ otherwise.} \end{cases} \qquad Z = \begin{cases} 0 \text{ if } X \text{ is even} \\ 1 \text{ otherwise.} \end{cases}$$

Find $I(Y; Z)$.

b. We roll a fair die which has six sides (opposite sides of a die add up to 7). What is the mutual information between the top side and the one facing you?

Exercise 5 (Entropy and Mutual Information)**.** Prove the following inequalities:

a. $H(X, Y|Z) \geq H(X|Z)$,

b. $I((X, Y); Z) \geq I(X; Z)$,

c. $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

Exercise 6 (Conditioning for mutual information)**.** Give examples of joint random variables X, Y, and Z such that

a. $I(X; Y|Z) < I(X; Y)$.

b. $I(X; Y|Z) > I(X; Y)$.

Exercise 7 (Entropy and pairwise independence)**.** Let X, Y, Z be three binary Bernoulli($\frac{1}{2}$) random variables that are pairwise independent; that is, $I(X; Y) = I(X; Z) = I(Y; Z) = 0$.

a. Under this constraint, what is the minimum value for $H(X, Y, Z)$?

b. Give an example achieving this minimum.

Exercise 8. (Conditioning and sub additivity) Prove the following.

a.

$$H(X_1, X_2, X_3) \leq \frac{1}{2} \left[H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1) \right].$$

b.

$$H(X_1, X_2, X_3) \geq \frac{1}{2} \left[H(X_1, X_2 | X_3) + H(X_2, X_3 | X_1) + H(X_3, X_1 | X_2) \right].$$

Exercise 9. Show that among all $\mathbb{N}$-valued random variables X with $\mathbb{E}[X] = \mu$, the $Geo(1/\mu)$ random variable has the maximum value of Shannon entropy.
Hint – Consider random variables X and Y with mean μ and taking values in $\mathbb{N}$ with $X \sim P_X$ and $Y \sim P_Y$ where P_Y is Geometric, and calculate $D(P_X || P_Y)$.

Exercise 10 (Conditional mutual information)**.** Consider a sequence of n binary random variables $X_1, X_2, \cdots, X_n$. Each sequence with an even number of 1's has probability $2^{-(n-1)}$, and each sequence with an odd number of 1's has probability 0. Find the mutual informations $I(X_1; X_2)$, $I(X_2; X_3 | X_1), \ldots, I(X_{n-1}; X_n | X_1, \ldots, X_{n-2})$.