ASSIGNMENT 3

Exercise 1. (Entropy of common distributions) Calculate the entropy of X where

a. X is the output of n independent tosses of a coin which shows heads with probability p.

b. X is a $Geo(p)$ random variable. That is, $\mathbb{P}[X = k] = (1-p)^{k-1}p$.

Exercise 2. (KL divergence) Calculate the KL divergence (relative entropy) between P and Q where

a. $P \equiv Geo(p)$ and $Q \equiv Geo(q)$.

b. $P \equiv \mathcal{N}(\mu_1, \sigma^2)$ and $Q \equiv \mathcal{N}(\mu_2, \sigma^2)$

Exercise 3. Show that among all $\mathbb{N}$-valued random variables X with $\mathbb{E}[X] = \mu$, the $Geo(1/\mu)$ random variable has the maximum value of Shannon entropy.
Hint – Consider random variables X and Y with mean μ and taking values in $\mathbb{N}$ with $X \sim P_X$ and $Y \sim P_Y$ where P_Y is Geometric, and calculate $D(P_X||P_Y)$.

Exercise 4 (Mutual information)**.** a. Let X be a uniform random variable over $\{1, 2, 3, 4\}$. Let

$$Y = \begin{cases} 0 \text{ if } X \text{ is odd} \\ 1 \text{ otherwise.} \end{cases} \qquad Z = \begin{cases} 0 \text{ if } X \text{ is even} \\ 1 \text{ otherwise.} \end{cases}$$

Find $I(Y; Z)$.

b. We roll a fair die which has six sides (opposite sides of a die add up to 7). What is the mutual information between the top side and the one facing you?

Exercise 5 (Entropy and Mutual Information)**.** Prove the following inequalities:

a. $H(X, Y|Z) \geq H(X|Z)$,

b. $I(X, Y; Z) \geq I(X; Z)$,

c. $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

Exercise 6 (Conditioning for mutual information)**.** Give examples of joint random variables X, Y, and Z such that

a. $I(X; Y|Z) < I(X; Y)$.

b. $I(X; Y|Z) > I(X; Y)$.

Exercise 7 (Entropy and pairwise independence)**.** Let X, Y, Z be three binary Bernoulli($\frac{1}{2}$) random variables that are pairwise independent; that is, $I(X; Y) = I(X; Z) = I(Y; Z) = 0$.

a. Under this constraint, what is the minimum value for $H(X, Y, Z)$?

b. Give an example achieving this minimum.

Exercise 8. (Shearer's lemma) Shearer's lemma is a generalization of the basic inequality

$$H(X_1, \ldots, X_n) \leq \sum_{i=1}^{n} H(X_i).$$

For $S \subseteq [n] = \{1, 2, \ldots\}$, we write $X_S = (X_i : i \in S)$.

a. Prove the lemma: Let $X_1, \ldots, X_n$ be random variables. Let $S_1, \ldots, S_m \subseteq [n]$ be subsets such that each $i \in [n]$ belongs to at least k sets. Then,

$$kH(X_1, \ldots, X_n) \leq \sum_{j=1}^{m} H(X_{S_j}).$$

Hint – Let $S_j = \{i_1, \ldots, i_{s_j}\}$ with $i_1 < \ldots < i_{s_j}$. Then,

$$\begin{aligned} H(X_{S_j}) &= H(X_{i_1}) + H(X_{i_2}|X_{i_1}) + \ldots + H(X_{i_{s_j}}|X_{i_1}, \ldots, X_{i_{s_j-1}}) \\ &\geq H(X_{i_1}|X_1, \ldots, X_{i_1-1}) + H(X_{i_2}|X_1, \ldots, X_{i_2-1}) + \ldots + H(X_{i_{s_j}}|X_1, \ldots, X_{i_{s_j}-1}). \end{aligned}$$

Sum the left side over $j = 1$ to m.

b. Suppose n distinct points in $\mathbb{R}^3$ have n_1 distinct projections on the XY-plane, n_2 distinct projections on the XZ-plane, and n_3 distinct projections on the YZ-plane. For two different points, since all three projections cannot be the same, we have $n \leq n_1 n_2 n_3$. Using Shearer's lemma, show that

$$n \leq \sqrt{n_1 n_2 n_3}.$$

Hint – Let $P = (X_1, X_2, X_3)$ be one of the n points picked uniformly at random. Then, $P_1 = (X_1, X_2)$, $P_2 = (X_1, X_3)$, and $P_3 = (X_2, X_3)$ are its three projections.